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Nonisospectral negative Volterra flows and mixed Volterra flows: Lax pairs, infinitely many conservation laws and integrable time discretization

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Abstract
In this paper, by means of the discrete zero curvature representation, nonisospectral negative Volterra flows and mixed Volterra flows are proposed. By means of solving corresponding discrete spectral equations, we demonstrate the existence of infinitely many conservation laws for the two nonisospectral flows and obtain the formulae of the corresponding conserved densities and associated fluxes. Integrable time discretizations for several isospectral equations of the two flows are also presented.

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1. Introduction

In recent years there has been wide interest in the study of nonlinear integrable lattice systems. It is well known that discrete lattice systems not only possess rich mathematical structures, but also have many applications in science, such as mathematical physics, numerical analysis, computer science, statistical physics, quantum physics, etc. Among the most famous and well-studied integrable lattices, the Volterra lattice

$$\dot{u}_n = u_n(u_{n+1} - u_{n-1}) \tag{1.1}$$

is one of the popular models. The Volterra lattice (1.1) has been studied extensively [1–6]. It relates to the following linear scattering problem:

$$E\psi(n,t,\lambda) = U_n\psi(n,t,\lambda) \qquad \frac{\mathrm{d}\psi(n,t,\lambda)}{\mathrm{d}t} = V_n\psi(n,t,\lambda) \tag{1.2}$$

where

$$U_n = \begin{pmatrix} \lambda & u_n \\ -1 & 0 \end{pmatrix} \qquad V_n = \begin{pmatrix} u_n & \lambda u_n \\ -\lambda & u_{n-1} - \lambda^2 \end{pmatrix}$$

and E is the shift operator in the variable n ($n \in Z$), defined by $E^j f(n, t, \lambda) = f(n + j, t, \lambda)$, and λ is the spectral parameter. Recently, Pritula and Vekslerchik [7] proposed the following isospectral negative Volterra flows:

$$\frac{\partial}{\partial t_{j+1}} \ln \frac{\tau_{n+1}}{\tau_{n-1}} + \frac{\tau_n^2}{\tau_{n-1}\tau_{n+1}} \frac{\partial^2}{\partial t_1 \partial t_j} \ln \tau_n = 0 \qquad j \geqslant 1 \tag{1.3}$$

and

$$\frac{\partial}{\partial t_1} \ln \frac{\tau_{n+1}}{\tau_{n-1}} = \frac{\tau_n^2}{\tau_{n-1}\tau_{n+1}}. \tag{1.4}$$

Here the field function u_n is expressed by the tau-function

$$u_n = \frac{\tau_{n+1}\tau_{n-2}}{\tau_n \tau_{n-1}}. \tag{1.5}$$

Dark-soliton solutions of negative Volterra flows (1.3) and (1.4) were constructed in [7]. The Bäcklund transformation and nonlinear superposition formula of the negative Volterra flows were obtained in [8]. As we know, the positive Volterra flows related to isospectral and nonisospectral problems have been constructed. Thus, a natural question is: what are nonisospectral negative Volterra flows? In this paper, we will first derive nonisospectral negative Volterra flows by means of the discrete zero curvature representation

$$\frac{\partial U_n}{\partial t} + \frac{\partial U_n}{\partial \lambda} \frac{d\lambda}{dt} = V_{n+1}^{(m)} U_n - U_n V_n^{(m)} \tag{1.6}$$

where matrix $V_n^{(m)}$ possesses negative powers of the spectral parameter λ. Then we will describe the nonisospectral mixed Volterra flows, which are a superposition of the nonisospectral positive and negative Volterra flows. It is well known that the existence of IMCL is a very important indicator of integrability of the system. From the viewpoint of physical and numerical analysis, it is also very interesting to know whether there exist conservation laws for a lattice system. In the present paper, by means of solving the corresponding discrete spectral equation which has been successfully applied for many isospectral lattice systems [9–13], we will demonstrate the existence of IMCL for the nonisospectral negative Volterra flows and mixed Volterra flows. The formulae of the corresponding conserved densities and associated fluxes will be derived. Thus, IMCL for the isospectral negative Volterra flows (1.3) and (1.4) will also be obtained. Another purpose of the present paper is to discuss integrable time discretization (difference–difference analogies) for the isospectral Volterra flows. Difference–difference analogies for the two flows of isospectral negative Volterra hierarchy and mixed Volterra hierarchy will be constructed.

2. Nonisospectral negative Volterra flows and mixed Volterra flows

In this section, we derive nonisospectral negative Volterra flows and mixed Volterra flows by means of the discrete zero curvature representation. Suppose the time evolution of the spectral parameter λ is described as

$$\frac{d\lambda}{dt} = a\lambda^{-(2m-1)} \qquad m \geqslant 1.$$

We construct the time evolution matrix $V_n^{(m)}$

$$V_n^{(m)} = \begin{pmatrix} A^{(m)}(n,t,\lambda) & -u_n E C^{(m)}(n,t,\lambda) \\ C^{(m)}(n,t,\lambda) & E^{-1}A^{(m)}(n,t,\lambda) + \lambda C^{(m)}(n,t,\lambda) \end{pmatrix} \tag{2.1}$$

where

$$A^{(m)}(n,t,\lambda) = \sum_{j=1}^{m} a_{m-j}(n,t)\lambda^{-2j} \qquad C^{(m)}(n,t,\lambda) = \sum_{j=1}^{m} c_{m-j}(n,t)\lambda^{-2j+1}$$

and $a_j(n,t), c_j(n,t)$ $(j = 0, 1, \ldots, m-1)$ are solutions to the following equations:

$$\begin{aligned} &(E - E^{-1})a_j(n,t) + (E-1)c_{j-1}(n,t) = 0 \qquad j = 1, 2, \ldots, m-1 \\ &(E-1)a_j(n,t) + u_{n+1}E^2 c_j(n,t) - u_n c_j(n,t) = 0 \qquad j = 1, 2, \ldots, m-1 \\ &(E - E^{-1})a_0(n,t) = 0 \qquad (E-1)a_0(n,t) + u_{n+1}E^2 c_0(n,t) - u_n c_0(n,t) = a. \end{aligned} \tag{2.2}$$

From the discrete zero curvature representation, we obtain the nonisospectral negative Volterra flows

$$\frac{\partial}{\partial t_m} u_n = u_n(E-1)c_{m-1}(n,t) \qquad m \geqslant 1 \tag{2.3}$$

which can be expressed in the τ_n function form,

$$\frac{\partial}{\partial t_m} \ln \frac{\tau_{n+1}}{\tau_{n-1}} = E c_{m-1}(n,t) \qquad m \geqslant 1 \tag{2.4}$$

where $c_{m-1}(n,t), m \geqslant 1$ has the formula:

$$\begin{aligned} &a_0(n,t) = 0 \qquad a_j(n,t) = \frac{\partial}{\partial t_j} \ln \frac{\tau_{n-1}}{\tau_n} \qquad j = 1, 2, \ldots, m-1 \\ &c_0(n,t) = \frac{\tau_{n-1}^2}{\tau_{n-2}\tau_n}\Delta_n \qquad c_j(n,t) = \frac{-\tau_{n-1}^2}{\tau_{n-2}\tau_n}\left(\frac{\partial^2}{\partial t_1 \partial t_j} \ln \tau_{n-1} - aF_n^{(j)}\right) \\ &\qquad j = 1, 2, \ldots, m-1 \end{aligned} \tag{2.5}$$

$$\Delta_n = 1 + a(E^2 - 1)^{-1}\left(\frac{\tau_n^2}{\tau_{n-1}\tau_{n+1}}\right) \tag{2.6}$$

$$F_n^{(j)} = \frac{\partial}{\partial t_j}(E^2-1)^{-1}\left[\frac{\tau_n^2}{\tau_{n-1}\tau_{n+1}}(E^2-1)^{-1}\left(\frac{\tau_{n+1}^2}{\tau_n \tau_{n+2}}\right)\right] \tag{2.7}$$

in which $(E^2-1)^{-1} = -\sum_{k=0}^{\infty} E^{2k}$. The first and second flows of the nonisospectral negative Volterra hierarchy are, respectively,

$$\frac{\partial}{\partial t_1} \ln \frac{\tau_{n+1}}{\tau_{n-1}} = \frac{\tau_n^2}{\tau_{n-1}\tau_{n+1}}\Delta_{n+1} \tag{2.8}$$

$$\frac{\partial}{\partial t_2} \ln \frac{\tau_{n+1}}{\tau_{n-1}} + \frac{\tau_n^2}{\tau_{n-1}\tau_{n+1}}\left(\frac{\partial^2}{\partial t_1^2} \ln \tau_n - aF_{n+1}^{(1)}\right) = 0. \tag{2.9}$$

If the time evolution of the spectral parameter λ is written as

$$\frac{\mathrm{d}\lambda}{\mathrm{d}t} = b\lambda^{2s+1} \qquad s \geqslant 0$$

and set the time evolution matrix

$$V_n^{(s)} = \begin{pmatrix} G^{(s)}(n,t,\lambda) & -u_n E H^{(s)}(n,t,\lambda) \\ H^{(s)}(n,t,\lambda) & E^{-1}G^{(s)}(n,t,\lambda) + \lambda H^{(s)}(n,t,\lambda) \end{pmatrix} \tag{2.10}$$

where

$$G^{(s)}(n,t,\lambda)=\sum_{j=0}^{s} g_{s-j}(n,t)\lambda^{2j} \qquad H^{(s)}(n,t,\lambda)=\sum_{j=0}^{s} h_{s-j}(n,t)\lambda^{2j+1}$$

and $g_j(n,t), h_j(n,t)$ $(j=0,1,\ldots,s)$ are solutions to the following equations:

$$\begin{aligned}&(E-E^{-1})g_{j-1}(n,t)+(E-1)h_j(n,t)=0 && j=1,2,\ldots,s\\ &(E-1)g_j(n,t)+u_{n+1}E^2h_j(n,t)-u_nh_j(n,t)=0 && j=1,2,\ldots,s\\ &(E-1)h_0(n,t)=0 \qquad (E-1)g_0(n,t)+u_{n+1}E^2h_0(n,t)-u_nh_0(n,t)=b\end{aligned} \tag{2.11}$$

then the nonisospectral positive Volterra flows are obtained,

$$\frac{\partial}{\partial t_s}u_n=u_n(E-E^{-1})g_s(n,t) \qquad s\geqslant 0 \tag{2.12}$$

where $g_s(n,t)$, $s\geqslant 0$ has the formula

$$\begin{aligned}&h_0(n,t)=-1 \qquad g_0(n,t)=u_n+nb \qquad h_1(n,t)=-(u_{n-1}+u_n+(2n-1)b)\\ &g_1(n,t)=u_n(u_{n-1}+u_n+u_{n+1}+(2n-1)b)+2b(E-1)^{-1}u_{n+1},\ldots.\end{aligned} \tag{2.13}$$

Here $(E-1)^{-1}=-\sum_{k=0}^{\infty}E^k$. Taking a superposition of nonisospectral positive and negative Volterra flows, we obtain the nonisospectral mixed Volterra flows and isospectral mixed Volterra flows corresponding to $\frac{d\lambda}{dt}=a\lambda^{-(2m-1)}$, $\frac{d\lambda}{dt}=b\lambda^{2s+1}$ and $\frac{d\lambda}{dt}=0$, respectively,

$$\frac{\partial}{\partial t_m}\ln\frac{\tau_{n+1}}{\tau_{n-1}}=Ec_{m-1}(n,t)+(E+1)\bar{g}_s(n,t) \qquad m\geqslant 1 \quad s\geqslant 0 \tag{2.14}$$

$$\frac{\partial}{\partial t_m}\ln\frac{\tau_{n+1}}{\tau_{n-1}}=E\bar{c}_{m-1}(n,t)+(E+1)g_s(n,t) \qquad m\geqslant 1 \quad s\geqslant 0 \tag{2.15}$$

$$\frac{\partial}{\partial t_m}\ln\frac{\tau_{n+1}}{\tau_{n-1}}=E\bar{c}_{m-1}(n,t)+(E+1)\bar{g}_s(n,t) \qquad m\geqslant 1 \quad s\geqslant 0 \tag{2.16}$$

where $\bar{c}_{m-1}(n,t)$ and $\bar{g}_s(n,t)$ are defined by $\bar{c}_{m-1}(n,t)=c_{m-1}(n,t)|_{a=0}$ and $\bar{g}_s(n,t)=g_s(n,t)|_{b=0}$. It is obvious that flows (2.14), (2.15) and (2.16) possess the Lax pairs U_n and $V_n^{(m,s)}$ in which

$$V_n^{(m,s)}=\begin{cases}V_n^{(m)}+V_n^{(s)}\big|_{b=0} & \text{for (2.14)}\\ V_n^{(m)}\big|_{a=0}+V_n^{(s)} & \text{for (2.15)}\\ V_n^{(m)}\big|_{a=0}+V_n^{(s)}\big|_{b=0} & \text{for (2.16).}\end{cases} \tag{2.17}$$

Example 1. Let $m=1, s=0$ in equations (2.14) and (2.15), we obtain two nonisospectral mixed Volterra lattice equations, respectively,

$$\frac{\partial}{\partial t_1}\ln\frac{\tau_{n+1}}{\tau_{n-1}}=\frac{\tau_n^2}{\tau_{n-1}\tau_{n+1}}\Delta_{n+1}+\frac{\tau_{n+1}\tau_{n-2}}{\tau_{n-1}\tau_n}+\frac{\tau_{n-1}\tau_{n+2}}{\tau_n\tau_{n+1}} \tag{2.18}$$

$$\frac{\partial}{\partial t_1}\ln\frac{\tau_{n+1}}{\tau_{n-1}}=\frac{\tau_n^2}{\tau_{n-1}\tau_{n+1}}+\frac{\tau_{n+1}\tau_{n-2}}{\tau_{n-1}\tau_n}+\frac{\tau_{n-1}\tau_{n+2}}{\tau_n\tau_{n+1}}+(2n+1)b. \tag{2.19}$$

Let $m=1, s=1$, another two nonisospectral mixed Volterra lattice equations are presented as follows:

$$\frac{\partial}{\partial t_1}\ln\frac{\tau_{n+1}}{\tau_{n-1}}=\frac{\tau_n^2}{\tau_{n-1}\tau_{n+1}}\Delta_{n+1}+(E+1)\left[\frac{\tau_{n+1}\tau_{n-3}}{\tau_{n-1}^2}+\frac{\tau_{n+2}\tau_{n-2}}{\tau_n^2}+\left(\frac{\tau_{n+1}\tau_{n-2}}{\tau_{n-1}\tau_n}\right)^2\right] \tag{2.20}$$

$$\frac{\partial}{\partial t_1}\ln\frac{\tau_{n+1}}{\tau_{n-1}}=\frac{\tau_n^2}{\tau_{n-1}\tau_{n+1}}+(E+1)\left[\frac{\tau_{n+1}\tau_{n-3}}{\tau_{n-1}^2}+\frac{\tau_{n+2}\tau_{n-2}}{\tau_n^2}+\left(\frac{\tau_{n+1}\tau_{n-2}}{\tau_{n-1}\tau_n}\right)^2\right]$$
$$+b(E+1)\left[(2n-1)\frac{\tau_{n+1}\tau_{n-2}}{\tau_{n-1}\tau_n}+2(E^2-1)^{-1}\frac{\tau_{n+2}\tau_{n-1}}{\tau_{n+1}\tau_n}\right]. \tag{2.21}$$

The mixed nonisospectral Volterra lattice equations corresponding to $m=2$, $s=0$ are

$$\frac{\partial}{\partial t_2}\ln\frac{\tau_{n+1}}{\tau_{n-1}}=\frac{-\tau_n^2}{\tau_{n-1}\tau_{n+1}}\left(\frac{\partial^2}{\partial t_1^2}\ln\tau_n-aF_{n+1}^{(1)}\right)+\frac{\tau_{n+1}\tau_{n-2}}{\tau_{n-1}\tau_n}+\frac{\tau_{n-1}\tau_{n+2}}{\tau_n\tau_{n+1}} \tag{2.22}$$

$$\frac{\partial}{\partial t_2}\ln\frac{\tau_{n+1}}{\tau_{n-1}}=\frac{-\tau_n^2}{\tau_{n-1}\tau_{n+1}}\frac{\partial^2}{\partial t_1^2}\ln\tau_n+\frac{\tau_{n+1}\tau_{n-2}}{\tau_{n-1}\tau_n}+\frac{\tau_{n-1}\tau_{n+2}}{\tau_n\tau_{n+1}}+(2n+1)b. \tag{2.23}$$

3. IMCL for nonisospectral negative Volterra flows and mixed Volterra flows

For a lattice equation

$$F(\dot q_n,\ddot q_n,\ldots,q_{n-1},q_n,q_{n+1},\ldots)=0 \tag{3.1}$$

if there exist functions ρ_n and J_n such that

$$\dot\rho_n|_{F=0}=J_n-J_{n+1} \tag{3.2}$$

then equation (3.2) is called the conservation law of equation (3.1), with ρ_n being the conserved density and J_n the associated flux. In this section, we first demonstrate the existence of IMCL for the lattice hierarchy related to nonisospectral problem (1.2) by means of solving the corresponding discrete spectral equations. Then we derive in detail IMCL for the nonisospectral negative Volterra lattice hierarchy and the mixed Volterra lattice hierarchy. The formulae of the corresponding conserved densities and associated fluxes are presented.

3.1. IMCL for lattice hierarchy associated with nonisospectral problem (1.2)

Consider the discrete spectral problem (1.2)

$$\psi_2(n+1,t,\lambda)=\lambda\psi_2(n,t,\lambda)-u_{n-1}\psi_2(n-1,t,\lambda) \tag{3.3}$$

which leads to a discrete Riccati-type equation,

$$u_{n-1}\Gamma_n\Gamma_{n+1}-\lambda\Gamma_{n+1}+1=0 \tag{3.4}$$

where $\Gamma_n=\frac{\psi_2(n-1,t,\lambda)}{\psi_2(n,t,\lambda)}$. Note that

$$\frac{(\psi_2(n+1,t,\lambda)\psi_2^{-1}(n,t,\lambda))_t}{\psi_2(n+1,t,\lambda)\psi_2^{-1}(n,t,\lambda)}=(E-1)\frac{(\psi_2(n,t,\lambda))_t}{\psi_2(n,t,\lambda)} \tag{3.5}$$

we obtain

$$\frac{\partial}{\partial t}[\ln(\lambda-u_{n-1}\Gamma_n)]=(E-1)Q_n \tag{3.6}$$

where

$$Q_n=V_{21}^{(m)}(n)(u_{n-1}\Gamma_n-\lambda)+V_{22}^{(m)}(n). \tag{3.7}$$

Suppose that the eigenfunction $\psi_2(n,t,\lambda)$ is the analytical function of the arguments, we obtain the Taylor series solution to equation (3.4),

$$\Gamma_n=\sum_{j=1}^{\infty}\lambda^{-j}w_n^{(j)} \tag{3.8}$$

where $w_n^{(j)}$ is presented recursively as follows,

$$w_n^{(2j)}=0 \qquad w_n^{(1)}=1 \qquad w_n^{(3)}=u_{n-2} \qquad w_n^{(5)}=u_{n-2}(u_{n-2}+u_{n-3})$$
$$w_n^{(2j+1)}=u_{n-2}\sum_{l+s=2j} w_{n-1}^{(l)}w_n^{(s)} \qquad j\geqslant 1. \tag{3.9}$$

From equation (3.6), we have

$$\frac{\partial}{\partial t}\left(-\ln\lambda+\sum_{k=1}^{\infty}\frac{\Phi^k}{k}\right)=(1-E)Q_n \tag{3.10}$$

where

$$\Phi=\lambda^{-1}u_{n-1}\Gamma_n=\sum_{j=1}^{\infty}\lambda^{-2j}u_{n-1}w_n^{(2j-1)}.$$

Furthermore, it follows from equation (3.10) that

$$-\lambda^{-1}\frac{\mathrm{d}\lambda}{\mathrm{d}t}+\frac{\partial}{\partial t}\sum_{j=1}^{\infty}\lambda^{-2j}\alpha_n^{(j)}$$
$$=-\lambda^{-1}\frac{\mathrm{d}\lambda}{\mathrm{d}t}-2\sum_{j=1}^{\infty}j\lambda^{-2j-1}\frac{\mathrm{d}\lambda}{\mathrm{d}t}\alpha_n^{(j)}+\sum_{j=1}^{\infty}\lambda^{-2j}\frac{\mathrm{d}\alpha_n^{(j)}}{\mathrm{d}t}=(1-E)Q_n \tag{3.11}$$

where

$$\alpha_n^{(j)}=v_{2j-1}+\frac{1}{2}\sum_{l_1+l_2=2j-2}v_{l_1}v_{l_2}+\frac{1}{3}\sum_{l_1+l_2+l_3=2j-3}v_{l_1}v_{l_2}v_{l_3}+\cdots$$
$$+\frac{1}{j-2}\sum_{l_1+l_2+\cdots+l_{j-2}=2j-j+2}v_{l_1}v_{l_2}\ldots v_{l_{j-2}}+v_1^{j-2}v_3+\frac{1}{j}v_1^j \tag{3.12}$$

in which $v_j=u_{n-1}w_n^{(j)}$. In comparison with the powers of λ on both sides of equation (3.11), we obtain IMCL for the lattice hierarchy related to nonisospectral equation (1.2)

$$\rho_{n,t}^{(j)}=(1-E)J_n^{(2j-1)} \qquad j\geqslant 1 \tag{3.13}$$

where $\rho_n^{(j)}$ ($j\geqslant 1$) corresponding to $\frac{\mathrm{d}\lambda}{\mathrm{d}t}=a\lambda^{-(2m-1)}$ and $\frac{\mathrm{d}\lambda}{\mathrm{d}t}=b\lambda^{2s+1}$ are presented as follows, respectively,

$$\rho_n^{(j)}=\begin{cases}\alpha_n^{(j)} & j=1,\ldots,m-1\\ \alpha_n^{(m)}-at & j=m\\ \alpha_n^{(j)}+2(m-j)a\int_0^t\alpha_n^{(j-m)}\,\mathrm{d}t & j\geqslant m+1\end{cases} \tag{3.14}$$

$$\rho_n^{(j)}=\alpha_n^{(j)}-2b(s+j)\int_0^t\alpha_n^{(s+j)}\,\mathrm{d}t \qquad j\geqslant 1. \tag{3.15}$$

Therefore, nonisospectral negative Volterra flows (2.3) and mixed Volterra flows (2.12) possess IMCL (3.13). Suppose that field function u_n is bounded for all n and is rapidly vanishing at infinity, then infinitely many conserved quantities H_i of nonisospectral lattice hierarchies (2.3) and (2.12) can be expressed in the following forms, respectively,

$$H_j=\sum_n\alpha_n^{(j)} \qquad j=1,2,\ldots,m-1$$
$$H_j=\sum_n\left(\alpha_n^{(j)}+2(m-j)a\int_0^t\alpha_n^{(j-m)}\,\mathrm{d}t\right) \qquad j\geqslant m+1 \tag{3.16}$$

and

$$H_j = \sum_n \left(\alpha_n^{(j)} - 2b(s+j) \int_0^t \alpha_n^{(s+j)}\, \mathrm{d}t \right) \qquad j \geqslant 1. \tag{3.17}$$

The integrand $\alpha_n^{(j)}$ in equations (3.16) and (3.17) is presented by equation (3.12). However, it should be noted that the sum of the conserved density $\rho_n^{(m)} = \alpha_n^{(m)} - at$ is not defined.

3.2. Flux functions for nonisospectral negative Volterra flows and mixed Volterra flows

In this subsection, we will derive the flux functions for nonisospectral negative Volterra flows and mixed Volterra flows. For nonisospectral negative Volterra lattice hierarchy (2.3), a direct computation leads to

$$Q_n = E^{-1} A^{(m)}(n) + u_{n-1} C^{(m)}(n) \Gamma_n = \sum_{j=1}^{\infty} J_n^{(j)} \lambda^{-2j} \tag{3.18}$$

where

$$J_n^{(j)} = \begin{cases} E^{-1} a_{m-j} + u_{n-1} \sum_{i=0}^{j-1} c_{m-j+i} w_n^{(2i+1)} & j = 1, \ldots, m \\ u_{n-1} \sum_{i=1}^{m} c_{m-i} w_n^{(2j-2i+1)} & j \geqslant m+1. \end{cases} \tag{3.19}$$

Thus, flux functions of equation (2.3) are presented by equation (3.19). For nonisospectral mixed Volterra lattice hierarchy (2.14), note that

$$Q_n = E^{-1}(A^{(m)}(n) + G^{(s)}(n)|_{b=0}) + u_{n-1}(C^{(m)}(n) + H^{(s)}(n)|_{b=0}) \Gamma_n = \sum_{j=1}^{\infty} J_n^{(j)} \lambda^{-2j} \tag{3.20}$$

where

$$J_n^{(j)} = \begin{cases} E^{-1} a_{m-j} + u_{n-1} \big(\sum_{i=0}^{j-1} c_{m-j+i} w_n^{(2i+1)} + \sum_{i=0}^{s} \bar{h}_{s-i} w_n^{(2j+2i+1)} \big) & j = 1, \ldots, m \\ u_{n-1} \big(\sum_{i=1}^{m} c_{m-i} w_n^{(2j-2i+1)} + \sum_{i=0}^{s} \bar{h}_{s-i} w_n^{(2j+2i+1)} \big) & j \geqslant m+1. \end{cases} \tag{3.21}$$

Therefore, flux functions of equation (2.14) are expressed in formula (3.21). For nonisospectral mixed Volterra lattice hierarchy (2.15), since

$$\begin{aligned} Q_n &= E^{-1}(A^{(m)}(n)|_{a=0} + G^{(s)}(n)) + u_{n-1}(C^{(m)}(n)|_{a=0} + H^{(s)}(n)) \Gamma_n \\ &= (n-1) b \lambda^{2s} + (E^{-1} g_1 + u_{n-1} h_1 - u_{n-1} u_{n-2}) \lambda^{2s-2} + \cdots \\ &\quad + E^{-1} g_s + u_{n-1} \sum_{j=0}^{s} h_{s-j} w_n^{(2j+1)} + \sum_{j=1}^{\infty} J_n^{(j)} \lambda^{-2j} \end{aligned} \tag{3.22}$$

where

$$J_n^{(j)} = \begin{cases} E^{-1} \bar{a}_{m-j} + u_{n-1} \big(\sum_{i=0}^{j-1} \bar{c}_{m-j+i} w_n^{(2i+1)} + \sum_{i=0}^{s} h_{s-i} w_n^{(2j+2i+1)} \big) & j = 1, \ldots, m \\ u_{n-1} \big(\sum_{i=1}^{m} \bar{c}_{m-i} w_n^{(2j-2i+1)} + \sum_{i=0}^{s} h_{s-i} w_n^{(2j+2i+1)} \big) & j \geqslant m+1 \end{cases} \tag{3.23}$$

we obtain flux functions formula for equation (2.15). For isospectral mixed Volterra lattice hierarchy (2.16), note that

$$Q_n = E^{-1}(A^{(m)}(n)|_{a=0} + G^{(s)}(n)|_{b=0}) + u_{n-1}(C^{(m)}(n)|_{a=0} + H^{(s)}(n)|_{b=0}) \Gamma_n. \tag{3.24}$$

By the formula of $\bar{g}_s(n)$ and $\bar{h}_s(n)$, we can prove that

$$Q_n = \sum_{j=1}^{\infty} J_n^{(j)} \lambda^{-2j} \tag{3.25}$$

where

$$J_n^{(j)} = \begin{cases} E^{-1}\bar{a}_{m-j} + u_{n-1}\left(\sum_{i=0}^{j-1} \bar{c}_{m-j+i} w_n^{(2i+1)} + \sum_{i=0}^{s} \bar{h}_{s-i} w_n^{(2j+2i+1)}\right) & j = 1, \ldots, m \\ u_{n-1}\left(\sum_{i=1}^{m} \bar{c}_{m-i} w_n^{(2j-2i+1)} + \sum_{i=0}^{s} \bar{h}_{s-i} w_n^{(2j+2i+1)}\right) & j \geqslant m+1. \end{cases} \tag{3.26}$$

Hence, flux functions of equation (2.16) have been obtained.

Example 2. The matrix $V_n^{(1)}$ of the first nonisospectral negative Volterra flow (2.8) has the form

$$V_n^{(1)} = \begin{pmatrix} 0 & -\frac{\tau_{n-2}\tau_n}{\lambda\tau_{n-1}^2}\Delta_{n+1} \\ \frac{\tau_{n-1}^2}{\lambda\tau_{n-2}\tau_n}\Delta_n & \frac{\tau_{n-1}^2}{\tau_{n-2}\tau_n}\Delta_n \end{pmatrix}.$$

Note that

$$Q_n = \sum_{j=1}^{\infty} J_n^{(2j-1)}\lambda^{-2j} \tag{3.27}$$

where

$$J_n^{(2j-1)} = \frac{\tau_{n-1}\tau_{n-3}}{\tau_{n-2}^2} w_n^{(2j-1)}\Delta_n \qquad j \geqslant 1. \tag{3.28}$$

Therefore the conserved densities $\rho_n^{(j)}$ and associated fluxes $J_n^{(2j-1)}$ for flow (2.8) are obtained.

Example 3. The matrix $V_n^{(2)}$ of the second nonisospectral negative Volterra flow (2.9) is written as

$$V_n^{(2)} = \begin{pmatrix} \frac{a_1}{\lambda^2} & -\frac{\tau_{n+1}\tau_{n-2}}{\tau_n\tau_{n-1}}\left(\frac{Ec_1}{\lambda} + \frac{\tau_n^2}{\lambda^3\tau_{n-1}\tau_{n+1}}\Delta_{n+1}\right) \\ \frac{c_1}{\lambda} + \frac{\tau_{n-1}^2}{\lambda^3\tau_{n-2}\tau_n}\Delta_n & c_1 + \frac{E^{-1}a_1}{\lambda^2} + \frac{\tau_{n-1}^2}{\lambda^2\tau_{n-2}\tau_n}\Delta_n \end{pmatrix}.$$

Note

$$\begin{aligned} Q_n &= \left(\frac{c_1}{\lambda} + \frac{\tau_{n-1}^2}{\lambda^3\tau_{n-2}\tau_n}\Delta_n\right)\left(\frac{\tau_n\tau_{n-3}}{\tau_{n-1}\tau_{n-2}}\Gamma_n - \lambda\right) + c_1 + \frac{E^{-1}a_1}{\lambda^2} + \frac{\tau_{n-1}^2}{\lambda^2\tau_{n-2}\tau_n}\Delta_n \\ &= \sum_{j=1}^{\infty} J_n^{(2j-1)}\lambda^{-2j} \end{aligned} \tag{3.29}$$

where

$$\begin{aligned} J_n^{(1)} &= E^{-1}a_1 + \frac{\tau_n\tau_{n-3}}{\tau_{n-1}\tau_{n-2}}c_1 \\ J_n^{(2j-1)} &= \frac{\tau_n\tau_{n-3}}{\tau_{n-2}\tau_{n-1}} w_n^{(2j-1)}c_1 + \frac{\tau_{n-1}\tau_{n-3}}{\tau_{n-2}^2} w_n^{(2j-3)}\Delta_n \qquad j \geqslant 2 \end{aligned} \tag{3.30}$$

we thus get its conserved densities $\rho_n^{(j)}$ and the associated fluxes $J_n^{(2j-1)}$ $(j \geqslant 1)$.

Example 4. The matrix $V_n^{(1,0)}$ of the nonisospectral mixed Volterra lattice equation (2.18) is

$$V_n^{(1,0)} = \begin{pmatrix} \frac{\tau_{n+1}\tau_{n-2}}{\tau_n\tau_{n-1}} & \frac{\lambda\tau_{n+1}\tau_{n-2}}{\tau_n\tau_{n-1}} - \frac{\tau_{n-2}\tau_n}{\lambda\tau_{n-1}^2}\Delta_{n+1} \\ \frac{\tau_{n-1}^2}{\lambda\tau_{n-2}\tau_n}\Delta_n - \lambda & \frac{\tau_n\tau_{n-3}}{\tau_{n-1}\tau_{n-2}} + \frac{\tau_{n-1}^2}{\tau_{n-2}\tau_n}\Delta_n - \lambda^2 \end{pmatrix}.$$

Note that

$$Q_n = \left[\frac{\tau_{n-1}^2}{\lambda\tau_{n-2}\tau_n}\Delta_n - \lambda\right]\left(\frac{\tau_n\tau_{n-3}}{\tau_{n-1}\tau_{n-2}}\Gamma_n - \lambda\right) + \frac{\tau_n\tau_{n-3}}{\tau_{n-1}\tau_{n-2}} + \frac{\tau_{n-1}^2}{\tau_{n-2}\tau_n}\Delta_n - \lambda^2$$
$$= \sum_{j=1}^{\infty} J_n^{(2j-1)}\lambda^{-2j} \tag{3.31}$$

where

$$J_n^{(2j-1)} = \frac{\tau_{n-1}\tau_{n-3}}{\tau_{n-2}^2} w_n^{(2j-1)}\Delta_n - \frac{\tau_n\tau_{n-3}}{\tau_{n-2}\tau_{n-1}} w_n^{(2j+1)} \qquad j \geqslant 1. \tag{3.32}$$

Therefore, we obtain the conserved densities $\rho_n^{(j)}$ and associated fluxes $J_n^{(2j-1)}$ for equation (2.18). For nonisospectral lattice equation (2.19), the matrix $V_n^{(1,0)}$ reads

$$V_n^{(1,0)} = \begin{pmatrix} \frac{\tau_{n+1}\tau_{n-2}}{\tau_n\tau_{n-1}} + nb & \frac{\lambda\tau_{n+1}\tau_{n-2}}{\tau_n\tau_{n-1}} - \frac{\tau_{n-2}\tau_n}{\lambda\tau_{n-1}^2} \\ \frac{\tau_{n-1}^2}{\lambda\tau_{n-2}\tau_n} - \lambda & \frac{\tau_n\tau_{n-3}}{\tau_{n-1}\tau_{n-2}} + \frac{\tau_{n-1}^2}{\tau_{n-2}\tau_n} - \lambda^2 + (n-1)b \end{pmatrix}.$$

Note that

$$Q_n = \left(\frac{\tau_{n-1}^2}{\lambda\tau_{n-2}\tau_n} - \lambda\right)\left(\frac{\tau_n\tau_{n-3}}{\tau_{n-1}\tau_{n-2}}\Gamma_n - \lambda\right) + \frac{\tau_n\tau_{n-3}}{\tau_{n-1}\tau_{n-2}} + \frac{\tau_{n-1}^2}{\tau_{n-2}\tau_n} - \lambda^2 + (n-1)b$$
$$= \sum_{j=1}^{\infty} J_n^{(2j-1)}\lambda^{-2j} \tag{3.33}$$

where

$$J_n^{(2j-1)} = \frac{\tau_{n-1}\tau_{n-3}}{\tau_{n-2}^2} w_n^{(2j-1)} - \frac{\tau_n\tau_{n-3}}{\tau_{n-2}\tau_{n-1}} w_n^{(2j+1)} \qquad j \geqslant 1. \tag{3.34}$$

Thus, the conserved densities $\rho_n^{(j)}$ and the associated fluxes $J_n^{(2j-1)}$ for equation (2.19) have been obtained.

Example 5. The matrix $V_n^{(1,1)}$ of the nonisospectral mixed Volterra lattice equation (2.20) is

$$V_n^{(1,1)} = \begin{pmatrix} g_1(n,t) + g_0(n,t)\lambda^2 & \frac{\tau_{n+1}\tau_{n-2}}{\tau_n\tau_{n-1}}(\lambda^3 - \lambda E h_1(n,t)) - \frac{\tau_{n-2}\tau_n}{\lambda\tau_{n-1}^2}\Delta_{n+1} \\ \frac{\tau_{n-1}^2}{\lambda\tau_{n-2}\tau_n}\Delta_n + h_1(n,t)\lambda - \lambda^3 & \frac{\tau_{n-1}^2}{\tau_{n-2}\tau_n}\Delta_n + E^{-1}g_1(n,t) - \frac{\tau_{n+1}\tau_{n-2}}{\tau_n\tau_{n-1}}\lambda^2 - \lambda^4 \end{pmatrix}$$

where

$$\begin{aligned} g_0(n,t) &= \frac{\tau_{n+1}\tau_{n-2}}{\tau_n\tau_{n-1}} \qquad h_1(n,t) = -\left(\frac{\tau_{n+1}\tau_{n-2}}{\tau_n\tau_{n-1}} + \frac{\tau_n\tau_{n-3}}{\tau_{n-1}\tau_{n-2}}\right) \\ g_1(n,t) &= \frac{\tau_{n+1}\tau_{n-3}}{\tau_{n-1}^2} + \frac{\tau_{n+2}\tau_{n-2}}{\tau_n^2} + \left(\frac{\tau_{n+1}\tau_{n-2}}{\tau_{n-1}\tau_n}\right)^2. \end{aligned} \tag{3.35}$$

Note that

$$Q_n = \left(\frac{\tau_{n-1}^2}{\lambda\tau_{n-2}\tau_n}\Delta_n + h_1(n,t)\lambda - \lambda^3\right)\left(\frac{\tau_n\tau_{n-3}}{\tau_{n-1}\tau_{n-2}}\Gamma_n - \lambda\right)$$
$$+ \frac{\tau_{n-1}^2}{\tau_{n-2}\tau_n}\Delta_n + E^{-1}g_1(n,t) - \frac{\tau_{n+1}\tau_{n-2}}{\tau_n\tau_{n-1}}\lambda^2 - \lambda^4$$
$$= \sum_{j=1}^{\infty} J_n^{(2j-1)}\lambda^{-2j} \tag{3.36}$$

where

$$J_n^{(2j-1)} = \frac{\tau_{n-1}\tau_{n-3}}{\tau_{n-2}^2} w_n^{(2j-1)} \Delta_n - \frac{\tau_n\tau_{n-3}}{\tau_{n-2}\tau_{n-1}} h_1(n,t) w_n^{(2j+1)} - \frac{\tau_n\tau_{n-3}}{\tau_{n-2}\tau_{n-1}} w_n^{(2j+3)} \qquad j \geqslant 1. \tag{3.37}$$

Hence the conserved densities $\rho_n^{(j)}$ and the associated fluxes $J_n^{(2j-1)}$ for equation (2.20) are obtained. For nonisospectral lattice equation (2.21), the matrix $V_n^{(1,1)}$ takes the form

$$V_n^{(1,1)} = \begin{pmatrix} g_1(n,t) + g_0(n,t)\lambda^2 & \frac{\tau_{n+1}\tau_{n-2}}{\tau_n\tau_{n-1}}(\lambda^3 - \lambda E h_1(n,t)) - \frac{\tau_{n-2}\tau_n}{\lambda\tau_{n-1}^2} \\ \frac{\tau_{n-1}^2}{\lambda\tau_{n-2}\tau_n} + h_1(n,t)\lambda - \lambda^3 & \frac{\tau_{n-1}^2}{\tau_{n-2}\tau_n} + E^{-1}g_1(n,t) - \left(\frac{\tau_{n+1}\tau_{n-2}}{\tau_n\tau_{n-1}} + nb\right)\lambda^2 - \lambda^4 \end{pmatrix}$$

where

$$g_0(n,t) = \frac{\tau_{n+1}\tau_{n-2}}{\tau_n\tau_{n-1}} + nb \qquad h_1(n,t) = -\left(\frac{\tau_{n+1}\tau_{n-2}}{\tau_n\tau_{n-1}} + \frac{\tau_n\tau_{n-3}}{\tau_{n-1}\tau_{n-2}} + (2n-1)b\right)$$

$$g_1(n,t) = \frac{\tau_{n+1}\tau_{n-3}}{\tau_{n-1}^2} + \frac{\tau_{n+2}\tau_{n-2}}{\tau_n^2} + \left(\frac{\tau_{n+1}\tau_{n-2}}{\tau_{n-1}\tau_n}\right)^2 + (2n-1)b\frac{\tau_{n+1}\tau_{n-2}}{\tau_{n-1}\tau_n} + 2b(E-1)^{-1}\frac{\tau_{n+2}\tau_{n-1}}{\tau_{n+1}\tau_n}.$$

Note that

$$\begin{aligned} Q_n &= \left(\frac{\tau_{n-1}^2}{\lambda\tau_{n-2}\tau_n} + h_1(n,t)\lambda - \lambda^3\right)\left(\frac{\tau_n\tau_{n-3}}{\tau_{n-1}\tau_{n-2}}\Gamma_n - \lambda\right) \\ &\quad + \frac{\tau_{n-1}^2}{\tau_{n-2}\tau_n} + E^{-1}g_1(n,t) - \left(\frac{\tau_{n+1}\tau_{n-2}}{\tau_n\tau_{n-1}} + nb\right)\lambda^2 - \lambda^4 \\ &= (n-1)b\lambda^2 - 2b\frac{\tau_n\tau_{n-3}}{\tau_{n-1}\tau_{n-2}} + 2b(E-1)^{-1}\frac{\tau_{n+1}\tau_{n-2}}{\tau_n\tau_{n-1}} + \sum_{j=1}^{\infty} J_n^{(2j-1)}\lambda^{-2j} \end{aligned} \tag{3.38}$$

where

$$J_n^{(2j-1)} = \frac{\tau_{n-1}\tau_{n-3}}{\tau_{n-2}^2} w_n^{(2j-1)} + \frac{\tau_n\tau_{n-3}}{\tau_{n-2}\tau_{n-1}} h_1(n,t) w_n^{(2j+1)} - \frac{\tau_n\tau_{n-3}}{\tau_{n-2}\tau_{n-1}} w_n^{(2j+3)} \qquad j \geqslant 1. \tag{3.39}$$

Therefore, we get conserved densities $\rho_n^{(j)}$ and associated fluxes $J_n^{(2j-1)}$ for equation (2.21).

4. Difference–difference analogies for the first flows of the isospectral negative Volterra flows and the mixed Volterra flows

Given an integrable lattice soliton equation, one would construct its difference–difference analogy. For many integrable lattice equations, such as the Toda lattice, the modified Toda lattice, the relativistic Toda lattice, the Bogoyavlensky lattice, the relativistic Volterra lattice, etc, difference–difference analogies have been obtained [14–20]. In this section, we establish the difference–difference analogies for the first flows of the isospectral negative Volterra flows and the mixed Volterra flows. Given a proper discrete spectral problem and its discrete-time evolution problem

$$E\psi(n,t,\lambda) = U_n\psi(n,t,\lambda) \qquad \tilde{\psi}(n,t,\lambda) = W_n\psi(n,t,\lambda) \tag{4.1}$$

where $\tilde{\psi}(n,t,\lambda) = \psi(n,t+h,\lambda)$. The compatibility of equation (4.1) leads to the following discrete zero curvature equation:

$$\tilde{U}_n W_n = W_{n+1} U_n. \tag{4.2}$$

If a difference–difference equation derived from equation (4.2) is a discrete-time approximation for a continuous-time lattice equation, then the difference equation is called the integrable discretization of the continuous-time lattice equation. In this case, the question is how do we choose a proper W_n. We do not have a general method to find such a matrix W_n. However, it should be noted that

$$\frac{\tilde{\psi}(n,t,\lambda)-\psi(n,t,\lambda)}{h}=\frac{(W_n-I)\psi(n,t,\lambda)}{h} \tag{4.3}$$

where I is the identity matrix. This means that

$$\lim_{h\to 0}\frac{W_n-I}{h}=V_n. \tag{4.4}$$

It is obvious that equation (4.4) is only a necessary condition in order to obtain difference–difference analogies. Now let us consider problem (4.1) in which

$$U_n=\begin{pmatrix}\lambda & u_n\\ -1 & 0\end{pmatrix}\qquad W_n=\begin{pmatrix}w_{11} & w_{12}\\ w_{21} & w_{22}\end{pmatrix}. \tag{4.5}$$

It follows from equation (4.2) that

$$w_{11}=Ew_{22}-\lambda Ew_{21} \tag{4.6}$$

$$w_{12}=-u_nEw_{21} \tag{4.7}$$

$$\lambda(E-1)w_{11}-Ew_{12}-\tilde{u}_nw_{21}=0 \tag{4.8}$$

$$\lambda w_{12}-u_nEw_{11}+\tilde{u}_nw_{22}=0. \tag{4.9}$$

Substituting equations (4.6) and (4.7) into equations (4.8) and (4.9), we have

$$\lambda^2(E^2-E)w_{21}+\lambda(E-E^2)w_{22}+\tilde{u}_nw_{21}-u_{n+1}E^2w_{21}=0 \tag{4.10}$$

$$\lambda u_n(E^2-E)w_{21}+\tilde{u}_nw_{22}-u_nE^2w_{22}=0. \tag{4.11}$$

For the first isospectral negative Volterra flow, let

$$w_{21}=\frac{hf(n)}{\lambda}\qquad w_{22}=1+\lambda w_{21} \tag{4.12}$$

where

$$\lim_{h\to 0}f(n)=c_0(n)=\frac{\tau_{n-1}^2}{\tau_{n-2}\tau_n}. \tag{4.13}$$

From equation (4.10), $f(n)$ satisfies the equation

$$\tilde{u}_nf(n)=u_{n+1}E^2f(n). \tag{4.14}$$

Note that $u_n=\frac{\tau_{n+1}\tau_{n-2}}{\tau_n\tau_{n-1}}$, we have

$$f(n)=\frac{\tilde{\tau}_{n-1}\tau_{n-1}}{\tilde{\tau}_{n-2}\tau_n}. \tag{4.15}$$

Therefore a difference–difference analogy of the first isospectral negative Volterra flow (1.4) is presented as follows:

$$\frac{\tilde{u}_n-u_n}{h}=u_nf(n+1)-\tilde{u}_nf(n). \tag{4.16}$$

Now let us express a difference–difference analogy for the first isospectral mixed Volterra flow:

$$\frac{\partial u_n}{\partial t}=u_n\left[u_{n+1}-u_{n-1}+(E-1)\frac{\tau_{n-1}^2}{\tau_{n-2}\tau_n}\right]. \tag{4.17}$$

Let

$$w_{11} = 1 + \frac{hP(n)}{1+\lambda^2 h} \qquad w_{21} = \frac{-\lambda h}{1+\lambda^2 h} + \frac{hQ(n)}{\lambda(1+\lambda^2 h)} \tag{4.18}$$

where

$$\lim_{h\to 0} P(n) = u_n \qquad \lim_{h\to 0} Q(n) = \frac{\tau_{n-1}^2}{\tau_{n-2}\tau_n}. \tag{4.19}$$

From equation (4.10), $P(n)$ and $Q(n)$ are written as

$$P(n) = (E-1)^{-1}(u_{n+1} - \tilde{u}_n) \qquad Q(n) = \frac{\tilde{\tau}_{n-1}\tau_{n-1}}{\tilde{\tau}_{n-2}\tau_n}. \tag{4.20}$$

We thus obtain a difference–difference analogy for equation (4.17),

$$\frac{\tilde{u}_n - u_n}{h} = u_n[P(n+1) + Q(n+1)] - \tilde{u}_n[P(n-1) + Q(n)]. \tag{4.21}$$

5. Conclusions

As is well known, the Lax pairs and IMCL are two important integrable properties for a discrete lattice system. The two integrable properties for many well-known lattice systems have been discussed. However, there is little work on IMCL for the lattice hierarchy with n-dependent coefficients in the literature. In this paper, by means of the discrete zero curvature representation, nonisospectral negative Volterra flows and mixed Volterra flows have been constructed. Furthermore, by means of solving discrete nonisospectral equations, we have demonstrated the existence of IMCL for the two nonisospectral flows and obtained the corresponding conserved densities and associated fluxes. Thus, their integrability has been further confirmed. To our knowledge, the explicit constructions of integrable lattice hierarchy and its infinitely many conserved quantities are remarkable in the case of hierarchies related to nonisospectral deformations of a linear spectral problem.

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